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Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Closed-Form Solution of Line-of-Sight Trajectory for Nonmaneuvering Targets

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I. Introduction

IN line-of-sight (LOS) trajectory, the pursuer always lies on the line between the target tracker and the target. If the pursuer is always on the tracker-target LOS, then the pursuer will surely hit the target. In other words, in LOS guidance, the pursuer maneuvers so as to be on the LOS between the target tracker and the target.[1–6]

A closed-form solution of a LOS trajectory for nonmaneuvering targets and a pursuer with constant velocity is not available. In 1955, Locke presented a 10-term series solution of this, for pursuer range vs pursuer angular position.[1] In addition, Macfadzean claimed in Ref. 3 that there is no closed-form solution for LOS trajectory, even with simplifying assumptions such as those made by Locke. In this Note, the closed-form solution of LOS trajectory for nonmaneuvering targets is derived.[6] Here, the total pursuer acceleration is assumed to be equal to the required acceleration in the direction normal to LOS, whereas Locke assumed the pursuer velocity is constant, that is, the pursuer acceleration is restricted in the direction normal to pursuer velocity.

II. Equations of Motion

The planar pursuer–target engagement geometry is shown in Fig. 1, where M and T are the pursuer and the target, respectively. Point O is fixed and represents the target tracker. Also, the OYZ coordinates system is nonrotating.

The equations of motion of particle p (M or T) in cylindrical coordinates (r, θ) according to Fig. 1 are as follows:

$$V_p = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta \tag{1}$$

$$a_p = (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\boldsymbol{e}_\theta \tag{2}$$

where $\boldsymbol{e}_r$ and $\boldsymbol{e}_\theta$ are unit vectors. If the pursuer is always on the LOS, then the pursuer will surely hit the target. Therefore, the basic guidance law is

$$\theta_m(t) = \theta_t(t) \tag{3}$$

where θ_m and θ_t are LOS angles from the tracker to the pursuer and to the target, respectively. We assume an ideal case in which the pursuer is always on the line between the target tracker and the target without any error. For this purpose, the required pursuer acceleration to put the pursuer on the LOS is

$$a_m = r_m\ddot{\theta}_t + 2\dot{r}_m\dot{\theta}_t \tag{4}$$

and the a_m is perpendicular to the LOS and its positive direction is shown in Fig. 1. Therefore, if the pursuer is initially fired at the target and accelerated according to Eq. (4), it will always lie on the tracker-target LOS. From now on, we will drop the subscript m or t for θ for convenience.

Here, we assume the nonmaneuvering target moves at constant altitude h with constant speed in the negative Y direction. Therefore, we have

$$r_t = h/\sin\theta \tag{5}$$

$$\cot\theta = \cot\theta_o - (V_t/h)t \tag{6}$$

$$\dot{\theta} = V_t \sin^2\theta/h \tag{7}$$

$$\ddot{\theta} = 2V_t^2 \sin^3\theta\cos\theta/h^2 \tag{8}$$

III. Solution for Nonmaneuvering Targets

We assume the total pursuer acceleration is equal to required acceleration; therefore, from Eq. (2), we have

$$\ddot{r}_m - r_m\dot{\theta}^2 = 0 \tag{9}$$

$$r_m\ddot{\theta} + 2\dot{r}_m\dot{\theta} = a_m \tag{10}$$

By using

$$\dot{r}_m = \dot{\theta}\frac{\mathrm{d}r_m}{\mathrm{d}\theta} \tag{11}$$

$$\ddot{r}_m = \ddot{\theta}\frac{\mathrm{d}r_m}{\mathrm{d}\theta} + \dot{\theta}^2\frac{\mathrm{d}^2r_m}{\mathrm{d}\theta^2} \tag{12}$$

we rewrite the differential equation (9) in the form of

$$\ddot{\theta}\frac{\mathrm{d}r_m}{\mathrm{d}\theta} + \dot{\theta}^2\left(\frac{\mathrm{d}^2r_m}{\mathrm{d}\theta^2} - r_m\right) = 0 \tag{13}$$

By the substituting from Eqs. (7) and (8) for $\dot{\theta}$ and $\ddot{\theta}$, the preceding differential equation simplifies to

$$\frac{\mathrm{d}^2r_m}{\mathrm{d}\theta^2} + 2\cot\theta\frac{\mathrm{d}r_m}{\mathrm{d}\theta} - r_m = 0 \tag{14}$$

By changing variable $u = r_m\sin\theta$, we can rewrite Eq. (14) as

$$\frac{1}{\sin\theta}\frac{\mathrm{d}^2u}{\mathrm{d}\theta^2} = 0 \tag{15}$$

which has a solution in the following form:

$$u = A_1\theta + A_2 \tag{16}$$

or

$$r_m = \frac{A_1\theta + A_2}{\sin\theta} \qquad \text{for} \qquad \theta_o \neq 0, \pi \tag{17}$$

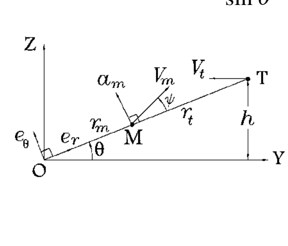

Fig. 1 Pursuer–target engagement geometry.

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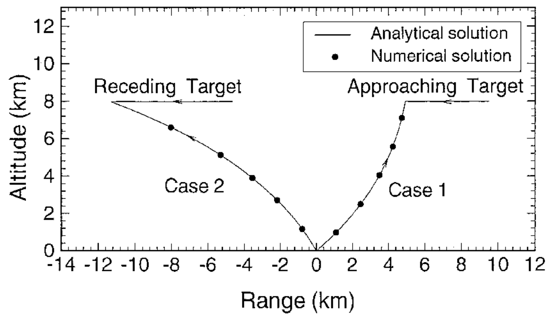
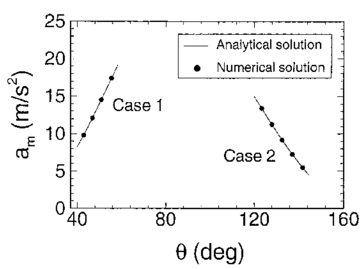


Fig. 2 Pursuer and target trajectories.

Fig. 3 Pursuer acceleration vs angular position.



where A_1 and A_2 are integrating constants and can be obtained from initial conditions. From Fig. 1, one can write

$$V_m \sin\psi = r_m \dot{\theta} \tag{18}$$

$$V_m \cos\psi = \dot{r}_m \tag{19}$$

Because in LOS guidance the pursuer is initially fired at the target from point O ($r_{m_o} = 0$), by using the preceding equations, we have $\psi_o = 0$ and $\dot{r}_{m_o} = V_{m_o}$; then by applying these initial conditions, the solution of differential equation of the pursuer motion is obtained as

$$r_m = \frac{hV_{m_o}}{V_t \sin\theta_o}\frac{(\theta - \theta_o)}{\sin\theta} \qquad \text{for} \qquad \theta_o \neq 0, \pi \tag{20}$$

Figure 2 shows the pursuer and the target trajectories for two cases. Case 1 is for an approaching target ($\theta_o = 40$ deg), and case 2 is for a receding target ($\theta_o = 120$ deg), and $V_{m_o} = 400$ m/s, $V_t = 200$ m/s, and $h = 8000$ m. To verify the analytical solution, the numerical solution of Eq. (9) is also obtained using a fourth-order Runge–Kutta method. The results in Fig. 2 show that analytical solution is correct.

By taking the time derivative of Eq. (20), $\dot{r}_m$ is obtained as follows:

$$\dot{r}_m = (V_{m_o}/\sin\theta_o)[\sin\theta - (\theta - \theta_o)\cos\theta] \tag{21}$$

Substituting from Eqs. (7) and (8) for $\dot{\theta}$ and $\ddot{\theta}$ and using Eqs. (20) and (21), we can obtain the pursuer acceleration from Eq. (4),

$$a_m = \frac{2V_{m_o}V_t}{h\sin\theta_o}\sin^3\theta \tag{22}$$

Figure 3 shows the pursuer acceleration vs angular position for two cases (similar to Fig. 2). The pursuer acceleration increases with time for an approaching target (case 1) and decreases for a receding target (case 2). The maximum pursuer acceleration occurs when the target is just above the target tracker, if the target passes above it. The symbols in Fig. 3 represent the numerical solution and is done to verify the analytical solution.

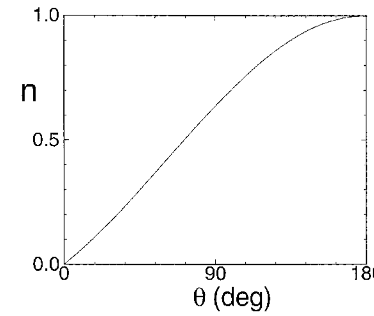
Dividing Eq. (19) by Eq. (18), we have

$$\cot\psi = \frac{1}{r_m}\frac{\mathrm{d}r_m}{\mathrm{d}\theta} \tag{23}$$

Therefore,

$$\tan\psi = \frac{\theta - \theta_o}{1 - (\theta - \theta_o)\cot\theta} \tag{24}$$

Fig. 4 Minimum value of *n* for interception.



Also, one can obtain

$$V_m^2 = \frac{V_{m_o}^2}{\sin^2\theta_o}\left[(\theta - \theta_o)^2 + \sin^2\theta - (\theta - \theta_o)\sin 2\theta\right] \tag{25}$$

The pursuer and the target positions are equal at the collision instant; therefore, by setting Eq. (5) equal to Eq. (20), we derive the final angular position:

$$\theta_f = \theta_o + \frac{V_t \sin\theta_o}{V_{m_o}} \tag{26}$$

Then, using Eq. (6), we obtain the final time as follows:

$$t_f = (h/V_t)(\cot\theta_o - \cot\theta_f) \tag{27}$$

Using Eqs. (5) and (20), for interception of the target, we must have

$$n > \sin\theta_o/(\pi - \theta_o) \qquad \text{for} \qquad \theta_o \neq 0, \pi \tag{28}$$

where n is the ratio of the pursuer initial velocity to the target velocity (V_{m_o}/V_t). Figure 4 shows the minimum value of n for interception.

The cumulative velocity increment is

$$\Delta V = \int_0^{t_f} |a_m|\,\mathrm{d}t \tag{29}$$

Then by using Eq. (22), we arrive at

$$\Delta V = (2V_{m_o}/\sin\theta_o)(\cos\theta_o - \cos\theta_f) \tag{30}$$

In the preceding derivation, we assumed planar engagement, but it is easy to extend the solution for three-dimensional engagement by appropriate coordinates transformation for nonmaneuvering targets. For this purpose, we pass the *YZ* plane from the target trajectory and point O, with the Y axis parallel to the target velocity vector in the opposite direction and with the X axis normal to the engagement plane. Therefore, we can use the same equations for three-dimensional pursuer–target engagement.

IV. Conclusions

The closed-form solution of the differential equations describing the LOS trajectory of the pursuer for nonmaneuvering targets is derived. Here, we assumed the total pursuer acceleration is equal to the required acceleration and the pursuer is always on the line between the target tracker and the target without any error. Also, some significant characteristics, such as total flight time, cumulative velocity increment, and the initial condition for interception, are investigated and discussed.

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